Editors Ilias Kotsireas Eugene Zima

COMPUTER 2006 Algebra 2006

Latest Advances in Symbolic Algorithms

Proceedings of the Waterloo Workshop in Computer Algebra 2006



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Editors

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Sergei Abramov

PREFACE

The Waterloo Workshop on Computer Algebra (WWCA-2006) was held on April 10–12, 2006 at Wilfrid Laurier University (Waterloo, Ontario, Canada) hosted by CARGO (http://www.cargo.wlu.ca). The workshop provided a forum for researchers and practitioners to discuss recent advances in the area of Computer Algebra. WWCA-2006 was dedicated to the 60th birthday of Sergei Abramov (Computer Center of the Russian Academy of Sciences, Moscow, Russia) whose influential contributions to symbolic methods are highly acknowledged by the research community and adopted by the leading Computer Algebra systems. The workshop attracted world-renowned experts from both the academia and the software industry. Presentations on original research topics or surveys of the state of the art advances in particular areas of Computer Algebra were made by

- Sergei Abramov, CCRAS, Russia
- Moulay Barkatou, University of Limoges, France
- Jacques Carette, McMaster University, Canada
- Robert Corless, University of Western Ontario, Canada
- Jürgen Gerhard, Maplesoft, Canada
- Oleg Golubitsky, Queens University, Canada
- Gaston Gonnet, ETH Zurich, Switzerland
- Kevin Hare, University of Waterloo, Canada
- Ilias Kotsireas, Wilfrid Laurier University, Canada
- George Labahn, University of Waterloo, Canada
- Ziming Li, Academy of Mathematics and System Sciences, China
- Luc Rebillard, University of Waterloo, Canada
- Bruno Salvy, INRIA Rocquencourt, France
- Éric Schost, University of Western Ontario, Canada
- Arne Storjohann, University of Waterloo, Canada
- Serguei Tsarev, Krasnoyarsk State Pedagogical University, Russia
- Mark van Hoeij, Florida State University, USA
- Thomas Wolf, Brock University, Canada
- Doron Zeilberger, Rutgers University, USA
- Eugene Zima, Wilfrid Laurier University, Canada

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Success of the workshop was also due to the support of the Academic Development Fund, Office of the Vice-President Academic, Research Office, and Department of Physics and Computer Science of Wilfrid Laurier University.

This book presents a collection of formally refereed selected papers submitted after workshop. Topics discussed in this book are the latest advances in algorithms of symbolic summation, factorization, symbolic-numeric linear algebra and linear functional equations, i.e. topics of symbolic computations that were extensively advanced due to Sergei's influencial works.

This book wouldn't have been possible without the contributions and hard work of the anonymous referees, who supplied detailed referee reports and helped authors improve their papers significantly.

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HYPERGEOMETRIC SUMMATION REVISITED

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We consider hypergeometric sequences, i.e., the sequences which satisfy linear first-order homogeneous recurrence equations with relatively prime polynomial coefficients. Some results related to necessary and sufficient conditions are discussed for validity of discrete Newton-Leibniz formula $\sum_{k=v}^{w} t(k) =$ u(w + 1) - u(v) when u(k) = R(k)t(k) and R(k) is a rational solution of Gosper's equation.

Keywords: Symbolic summation; Hypergeometric sequences; Discrete Newton-Leibniz formula.

1. Introduction

Let K be a field of characteristic zero $(K = \mathbb{C} \text{ in all examples})$. If $t(k) \in K(k)$ then the *telescoping equation*

$$u(k+1) - u(k) = t(k)$$
(1)

may or may not have a rational solution u(k), depending on the type of t(k). Here the telescoping equation is considered as an equality in the rationalfunction field, regardless of the possible integer poles that u(k) and/or t(k)might have.

An algorithm for finding rational u(k) was proposed in 1971 (see Ref. 1). It follows from that algorithm that if t(k) has no integer poles, then a

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[†]Partially supported by MVZT RS under grant P1-0294.

rational u(k) satisfying (1), if it exists, has no integer poles either, and the discrete Newton-Leibniz formula

$$\sum_{k=v}^{w} t(k) = u(w+1) - u(v)$$
(2)

is valid for any integer bounds $v \leq w$. Working with polynomial and rational functions we will write $f(k) \perp g(k)$ for $f(k), g(k) \in K[k]$ to indicate that f(k) and g(k) are coprime; if $R(k) \in K(k)$, then den(R(k))is the monic polynomial from K[k] such that $R(k) = \frac{f(k)}{den(R(k))}$ for some $f(k) \in K[k], f(k) \perp den(R(k))$.

The problem of solving equation (1) can be considered for sequences. If t(k) is a sequence, we use the symbol E for the shift operator w.r. to k, so that Et(k) = t(k + 1). In the rest of the paper we assume that the sequences under consideration are defined on an infinite interval I of integers and either $I = \mathbb{Z}$, or

$$I = \mathbb{Z}_{\geq l} = \{k \in \mathbb{Z} \mid k \geq l\}, \ l \in \mathbb{Z}.$$

If a sequence t(k) defined on I is given, and a sequence u(k), which is also defined on I and satisfies (1) for all $k \in I$, is found (any such sequence is a *primitive* of t(k)), then we can use formula (2) for any $v \leq w$ with $v, w \in I$.

Gosper's algorithm,⁶ which we denote hereafter by \mathcal{GA} , discovered in 1978, focuses on the case where a given t(k) and an unknown u(k) are hypergeometric sequences.

Definition 1.1. A sequence y(k) defined on an infinite interval I is hypergeometric if it satisfies the equation Ly(k) = 0 for all $k \in I$, with

$$L = a_1(k)E + a_0(k) \in K[k, E], \quad a_1(k) \perp a_0(k).$$
(3)

 \mathcal{GA} starts by constructing the operator L for a given concrete hypergeometric sequence t(k), and this step is not formalized. On the next steps \mathcal{GA} works with L only, while the sequence t(k) itself is ignored (more precisely, in the case of $L = a_1(k)E + a_0(k)$, \mathcal{GA} works with the *certificate* of t(k), i.e., with the rational function $-\frac{a_0(k)}{a_1(k)}$, but this is not essential). The algorithm tries to construct a rational function R(k), which is a solution in K(k) of *Gosper's equation*

$$a_0(k)R(k+1) + a_1(k)R(k) = -a_1(k)$$
(4)

(such R(k), when it exists, can also be found by general algorithms from Refs. 2,3). If such R(k) exists then

$$R(k+1)t(k+1) - R(k)t(k) = t(k)$$

is valid for *almost* all integers k. The fact is that even when t(k) is defined everywhere on I, it can happen that R(k) has some poles belonging to I, and u(k) = R(k)t(k) cannot be defined in such a way as to make (1) valid for all integers from I. One can encounter the situation where formula (2) is not valid even when all of

$$t(v), t(v+1), \dots, t(w), u(v), u(w+1)$$

are well-defined. The reason is that (1) may fail to hold at certain points k of the summation interval. However, sometimes it is possible to define the values of u(k) = R(k)t(k) appropriately for all integers k, even though R(k) has some integer poles. In such well-behaved cases (2) can be used to compute $\sum_{k=v}^{w} t(k)$ for any $v \leq w, v, w \in I$.

Example 1.1.

Gosper's equation, corresponding to $L = kE - (k+1)^2$, has a solution $R = \frac{1}{k}$. The sequences

$$t_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ k \cdot k!, & \text{if } k \ge 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} \frac{(-1)^k k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \ge 0 \end{cases}$$

both satisfy Ly = 0 on $I = \mathbb{Z}$.

Generally speaking, (2) is not applicable to $t_1(k)$, but is applicable to $t_2(k)$. We can illustrate this as follows. Applying (2) to $t_1(k)$ with v = -1, w = 1, we have

$$t_1(-1) + t_1(0) + t_1(1) = \frac{1}{k}t_1(k)|_{k=2} - \frac{1}{k}t_1(k)|_{k=-1} = \frac{1}{2} \cdot 4 - 0 = 2$$

which is wrong, because $t_1(-1) + t_1(0) + t_1(1) = 0 + 0 + 1 = 1$. Applying (2) to t_2 with the same v, w, we have

$$t_2(-1) + t_2(0) + t_2(1) = \frac{1}{k}t_2(k)|_{k=2} - \frac{1}{k}t_2(k)|_{k=-1} = 0 - (-1) = 1$$

which is correct, because $t_2(-1) + t_2(0) + t_2(1) = 1 + 0 + 0 = 1$.

In this paper we discuss some results related to necessary and sufficient conditions for validity of formula (2) when u(k) = R(k)t(k), and R(k) is a rational solution of corresponding Gosper's equation. If such R(k) exists, then we describe the linear space of all hypergeometric sequences t(k) that are defined on I and such that formula (2) is valid for u = Rt and any integer bounds $v \leq w$ such that $v, w \in I$. The dimension of this space is always positive (it can be even bigger than 1). We will denote

- by \mathcal{H}_I the set of all hypergeometric sequences defined on I;
- by \mathcal{L} the set of all operators of type (3);
- by $V_I(L)$, where $L \in \mathcal{L}$, the K-linear space of all sequences t(k) defined on I for which Lt(k) = 0 for all $k \in I$;
- by $W_I(R(k), L)$, where $L \in \mathcal{L}$ and $R(k) \in K(k)$ is a solution of the corresponding Gosper's equation, the K-linear space of all $t(k) \in V_I(L)$ such that (2) with u(k) = R(k)t(k) is valid for all $v \leq w$ with $v, w \in I$.

The paper is a summary of the results that have been published in Refs. 4,5. In addition we consider the case where Gosper's equation has non-unique rational solution (Section 3.2). In Section 2 we consider individual hypergeometric sequences while in Section 3 we concentrate on spaces of the type $W_I(R(k), L)$.

2. Validity conditions of the discrete Newton-Leibniz formula

2.1. A criterion

Theorem 2.1.^{4,5} Let $L \in \mathcal{L}$, $t(k) \in V_I(L)$, and let Gosper's equation corresponding to L have a solution $R(k) \in K(k)$, with den(R) = g(k). Then $t(k) \in W_I(R(k), L)$ iff there exists a $\overline{t}(k) \in \mathcal{H}_I$ such that $t(k) = g(k)\overline{t}(k)$ for all $k \in I$.

Example 2.1. Consider again the sequences $t_1(k), t_2(k)$ on $I = \mathbb{Z}$ from Example 1.1. We have $t_2(k) = k\bar{t}_2(k)$, where

$$\bar{t}_2(k) = \begin{cases} \frac{(-1)^k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \ge 0 \end{cases}$$

is a hypergeometric sequence defined everywhere:

$$E\bar{t}_2(k) - (k+1)\bar{t}_2(k) = 0.$$

On the other hand, if $t_1(k) = k\bar{t}_1(k)$ for some sequence $\bar{t}_1(k)$, then

$$\bar{t}_1(k) = \begin{cases} 0, \text{ if } k < 0, \\ \zeta, \text{ if } k = 0, \\ k!, \text{ if } k > 0 \end{cases}$$

where $\zeta \in \mathbb{C}$. Notice that the sequence $\bar{t}_1(k)$ is not hypergeometric on \mathbb{Z} , for any $\zeta \in \mathbb{C}$.

2.2. Summation of proper hypergeometric sequences

Definition 2.1. Following conventional notation, the rising factorial power $(\alpha)_k$ and its reciprocal $1/(\beta)_k$ are defined for $\alpha, \beta \in K$ and $k \in \mathbb{Z}$ by

$$(\alpha)_k = \begin{cases} \prod_{m=0}^{k-1} (\alpha+m), k \ge 0; \\ \prod_{m=1}^{|k|} \frac{1}{\alpha-m}, k < 0, \ \alpha \ne 1, 2, \dots, |k|; \\ \text{undefined}, \text{ otherwise}; \end{cases}$$

$$\frac{1}{(\beta)_k} = \begin{cases} \prod_{m=0}^{k-1} \frac{1}{\beta+m}, & k \ge 0, \ \beta \ne 0, -1, \dots, 1-k; \\ \prod_{\substack{|k| \\ m=1 \\ \text{undefined}, \\ m = 1 \\ \text{otherwise.}} \end{cases}$$

Note that if $(\alpha)_k$ resp. $1/(\beta)_k$ is defined for some $k \in \mathbb{Z}$, then $(\alpha)_{k+1}$ resp. $1/(\beta)_{k-1}$ is defined for that k as well. Thus $(\alpha)_k$ and $1/(\beta)_k$ are hypergeometric sequences which satisfy

$$(\alpha)_{k+1} = (\alpha+k)(\alpha)_k, \quad (\beta+k)/(\beta)_{k+1} = 1/(\beta)_k$$
 (5)

whenever $(\alpha)_k$ and $1/(\beta)_{k+1}$ are defined.

Example 2.2. Let $t(k) = (k-2)(-1/2)_k/(4k!)$. This hypergeometric sequence is defined for all $k \in \mathbb{Z}$ (note that t(k) = 0 for k < 0) and satisfies Lt(k) = 0 for all $n \in \mathbb{Z}$ where $L = a_1(k)E + a_0(k)$ with $a_0(k) = -(k-1)(2k-1)$ and $a_1(k) = 2(k-2)(k+1)$. Gosper's equation, corresponding to L, has a rational solution

$$R(k) = \frac{2k(k+1)}{k-2}.$$
 (6)

Equation (1) indeed fails at k = 1 and k = 2 because u(k) = R(k)t(k) is undefined at k = 2. But if we cancel the factor k - 2 and replace u(k) by the sequence

$$\bar{u}(k) = k(k+1)\frac{(-1/2)_k}{2k!},$$

then equation

$$\bar{u}(k+1) - \bar{u}(k) = t(k)$$
 (7)

holds for all $k \in \mathbb{Z}$, and

$$\sum_{k=v}^{w} t(k) = \bar{u}(w+1) - \bar{u}(v).$$
(8)

The sequence t(k) from Example 2.2 is an instance of a proper hypergeometric sequence which we are going to define now. As it turns out, there are no restrictions on the validity of the discrete Newton-Leibniz formula for proper sequences (Theorem 2.2).

Definition 2.2. A hypergeometric sequence t(k) defined on an infinite interval I of integers is *proper* if there are

- a constant $z \in K$,
- a polynomial $p(k) \in K[k]$,
- nonnegative integers q, r,
- constants $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_r \in K$

such that

$$t(k) = p(k)z^{k} \frac{\prod_{i=1}^{q} (\alpha_{i})_{k}}{\prod_{j=1}^{r} (\beta_{j})_{k}}$$
(9)

for all $k \in I$.

Theorem 2.2.⁴ Let t(k) be a proper hypergeometric sequence defined on Iand given by (9). Denote $a(k) = z \prod_{i=1}^{q} (k + \alpha_i)$ and $b(k) = \prod_{j=1}^{r} (k + \beta_j)$. If a polynomial $y(k) \in K[k]$ satisfies

$$a(k)y(k+1) - b(k-1)y(k) = p(k)$$
(10)

and if

$$\bar{u}(k) = y(k)z^k \frac{\prod_{i=1}^q (\alpha_i)_k}{\prod_{j=1}^r (\beta_j)_{k-1}}$$

for all $k \in I$, then equation (7) holds for all $k \in I$, and the discrete Newton-Leibniz formula (8) is valid for all $v \leq w$, when $v, w \in I$.

Notice that (10) has a solution in K[k] iff Gosper's equation, corresponding to the operator from \mathcal{L} , annihilating t(k), has a solution in K(k).

Example 2.3. The hypergeometric sequence

$$t(k) = \frac{\binom{2k-3}{k}}{4^k},$$
 (11)

which is defined for all $k \in \mathbb{Z}$ can be written as

$$t(k) = \begin{cases} 2s(k), \ k < 2\\ s(k), \ k \ge 2 \end{cases}$$

where

$$s(k) = (2-k)\frac{(-1/2)_k}{4(1)_k}$$

is the proper sequence from Example 2.2. For $w \ge 1$, one should first split summation range in two

$$\sum_{k=0}^{w} t(k) = \frac{3}{4} + \sum_{k=2}^{w} s(k),$$

then the discrete Newton-Leibniz formula can be safely used to evaluate the sum on the right. However, applying directly (2) to (11) with (6) we obtain

$$\sum_{k=0}^{w} t(k) = (?) \quad u(w+1) - u(0) = \frac{(w+1)(w+2)\binom{2w-1}{w+1}}{2(w-1)4^{w}}.$$
 (12)

If we assume that the value of $\binom{2k-3}{k}$ is 1 when k = 0 and -1 when k = 1 (that is natural from combinatorial point of view) then the expression on the right gives the true value of the sum only at w = 0.

2.3. When the interval I contains no leading integer singularity of L

Definition 2.3. For a linear difference operator (3) we call $M = \max(\{k \in \mathbb{Z}; a_1(k-1) = 0\} \cup \{-\infty\})$ the maximal leading integer singularity of L,

Proposition 2.1.⁴ Let R(k) be a rational solution of (4). Then R(k) has no poles larger than M - 1.

Theorem 2.3.⁴ Let $L \in \mathcal{L}$, M be the maximal integer singularity of L, $l \geq M$, $I = \mathbb{Z}_{\geq l}$ and $t(k) \in V_I(L)$. Let Gosper's equation, corresponding to L, have a solution R(k) in K(k). Then $t(k) \in W_I(R(k), L)$.

Example 2.4. For the sequence (11) we have $a_0(k) = -(2k-1)(k-1)$, $a_1(k) = 2(k+1)(k-2)$, R(k) = 2k(k+1)/(k-2), and $u(k) = 2k(k+1)\binom{2k-3}{k}/((k-2)4^k)$. Thus M = 3, and the only pole of R(k) is k = 2. As predicted by Theorem 2.3, the discrete Newton-Leibniz formula is valid when, e.g., $3 \le v \le w$.

3. The spaces $V_I(L)$ and $W_I(R(k), L)$

3.1. The structure of $W_I(R(k), L)$

Theorem 3.1.⁵ Let $L \in \mathcal{L}$ and Gosper's equation, corresponding to L, have a solution $R(k) \in K(k)$, den(R) = g(k). Then

$$W_I(R(k), L) = g(k) \cdot V_I(\operatorname{pp}(L \circ g(k))),$$

where the operator $pp(L \circ g(k))$ is computed by removing from $L \circ g$ the greatest common polynomial factor of its coefficients.

In addition, if $R = \frac{f(k)}{g(k)}$, $f(k) \perp g(k)$, then the space of the corresponding primitives of the elements of $W_I(R(k), L)$ can be described as $f(k) \cdot V_I(\text{pp}(L \circ g(k)))$.

We will denote by \overline{L} the operator $pp(L \circ g(k))$.

Example 3.1. Consider again the operator $L = kE - (k+1)^2$ from Example 1.1 with $I = \mathbb{Z}$. We have $R = \frac{1}{k}$, and

$$L \circ k = kE \circ k - (k+1)^2 k = k(k+1)E - (k+1)^2 k = k(k+1)(E-k-1),$$

$$\bar{L} = E - (k+1).$$

The space $W_I(R(k), \bar{L})$ is generated by \bar{t}_2 , and, resp., the space $k \cdot W_I(R(k), \bar{L})$ is generated by $k\bar{t}_2$. In accordance with Theorem 3.1 the space $W_I(R(k), L)$ coincides with $k \cdot V_I(\bar{L})$.

It is possible to give examples showing that in some cases $\dim W_I(R(k), L) > 1$.

Example 3.2.

Let $L = 2(k^2-4)(k-9)E - (2k-3)(k-1)(k-8)$, $I = \mathbb{Z}$. Then Gosper's equation, corresponding to L, has the rational solution

$$R(k) = -\frac{2(k-3)(k+1)}{k-9}.$$

Here g(k) = k - 9 and $\overline{L} = 2(k^2 - 4)E - (2k - 3)(k - 1)$. Any sequence \overline{t} which satisfies the equation $\overline{L}\overline{t} = 0$ has $\overline{t}(k) = 0$ for k = 2 or $k \leq -2$. The

values of $\bar{t}(1)$ and $\bar{t}(3)$ can be chosen arbitrarily, and all the other values are determined uniquely by the recurrence $2(k^2-4)\bar{t}(k+1) = (2k-3)(k-1)\bar{t}(k)$. Hence dim $V_I(\bar{L}) = 2$.

At the same time, dim $V_I(L) = 3$. Indeed, if Lt = 0, then t(-2) = t(2) = t(9) = 0. The value t(k) = 0 from k = -2 propagates to all $k \leq -2$, but on each of the integer intervals [-1, 0, 1], [3, 4, 5, 6, 7, 8] and [10, 11, ...) we can choose one value arbitrarily, and the remaining values on that interval are then determined uniquely. A sequence $t(k) \in V_I(L)$ belongs to $W_I(R(k), L)$ iff 22t(10) - 13t(8) = 0. So dim $W_I(R(k), L) = 2$.

3.2. When a rational solution of Gosper's equation is not unique

We give an example showing that if $L \in \mathcal{L}$ and Gosper's equation, corresponding to L, has different solutions $R_1(k), R_2(k) \in K(k)$, then it is possible that $W_I(R_1(k), L) \neq W_I(R_2(k), L)$. Moreover, these two spaces can have different dimensions.

Example 3.3. If L = kE - (k+1), then Gosper's equation, corresponding to L, is

$$-(k+1)R(k+1) + kR(k) = -k,$$

and its general rational solution is

$$\frac{k-1}{2} + \frac{c}{k} = \frac{k^2 - k + 2c}{2k}.$$

Consider the solutions

$$R_1(k) = \frac{k-1}{2} (g_1(k) = 1), \text{ and } R_2(k) = \frac{k^2 - k + 2}{2k} (g_2(k) = k).$$

We have $L \circ g_1(k) = L$, and $W_I(R_1(k), L) = V_I(L)$. This space has a basis that consists of two linearly independent sequences:

$$t_1(k) = \begin{cases} k, \text{ if } k \le 0, \\ 0, \text{ if } k > 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} 0, \text{ if } k \le 0, \\ k, \text{ if } k > 0. \end{cases}$$

So this space contains, e.g., the sequence t(k) = |k|.

We have $L \circ g_2(k) = k(k+1)(E-1)$, therefore $W_I(R_2(k), L)$ is generated by the sequence t(k) = k. If Gosper's equation, corresponding to $L \in \mathcal{L}$, has non-unique solution in K(k), then the equation Ly = 0 has a non-zero solution in K(k).

3.3. If Gosper's equation has a rational solution R(k) then $W_I(R,L) \neq 0$

Theorem 3.2.⁵ Let $L \in \mathcal{L}$ and let Gosper's equation, corresponding to L, have a solution $R(k) \in K(k)$. Then $W_I(R(k), L) \neq 0$ (i.e., $\dim W_I(R(k), L) \geq 1$).

Example 3.4.

Let L = (k+2)E - k. The rational function $\frac{1}{k(k+1)}$ is a solution in K(k) of the equation Ly = 0. Here R(k) = -k - 1, and -1/k is a solution of the corresponding telescoping equation:

$$-\frac{1}{k+1} + \frac{1}{k} = \frac{1}{k(k+1)}.$$

The rational functions

$$\frac{1}{k(k+1)}$$
 and $-\frac{1}{k}$

have integer poles. Nevertheless, by Theorem 3.2 it has to be $W_I(R(k), L) \neq 0$ even when $I = \mathbb{Z}$. The space $W_I(R(k), L)$ is generated by the sequence

$$t(k) = \begin{cases} 1, & \text{if } k = -1, \\ -1, & \text{if } k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

while the primitive of t(k) is

$$(-k-1)t(k) = \begin{cases} 1, \text{ if } k = 0, \\ 0, \text{ otherwise} \end{cases}$$

If $I = \mathbb{Z}_{\geq 1}$, then $W_I(R(k), L)$ is generated by the sequence $t'(k) = \frac{1}{k(k+1)}$.

By Theorem 2.3, if M is the maximal integer singularity of L, $l \ge M$, $I = \mathbb{Z}_{\ge l}$, and Gosper's equation, corresponding to L, has a solution R(k) in K(k), then $V_I(L) = W_I(R(k), L)$. As a consequence, dim $V_I(L) = \dim W_I(R(k), L) = 1$.

References

 S. A. Abramov, On the summation of rational function, USSR Comput. Math. Phys. 11 (1971), 324–330. Transl. from Zh. vychisl. mat. mat. fyz. 11 (1971), 1071–1075.

- S. A. Abramov, Rational solutions of linear difference and differential equations with polynomial coefficients, USSR Comput. Math. Phys. 29 (1989), 7–12. Transl. from Zh. vychisl. mat. mat. fyz. 29 (1989), 1611–1620.
- S. A. Abramov, Rational solutions of linear difference and q-difference equations with polynomial coefficients, *Programming and Comput. Software* 21 (1995), 273–278. Transl. from *Programmirovanie* 21 (1995), 3–11.
- S. A. Abramov and M. Petkovšek, Gosper's Algorithm, Accurate Summation, and the discrete Newton-Leibniz formula, ISSAC'05 (Annual International Symposium on Symbolic and Algebraic Computation). Beijing, China; 24– 27 July 2005; *Proceedings ISSAC'05*, 5–12.
- S. A. Abramov, On the summation of P-recursive sequences, ISSAC'06 (Annual International Symposium on Symbolic and Algebraic Computation). Genova, Italy; 9–12 July 2006; Proceedings ISSAC'06, 17–22.
- R. W. Gosper, Jr., Decision procedure for indefinite hypergeometric summation, Proc. Natl. Acad. Sci. USA 75 (1978), 40–42.