INDUCTANCE CALCULATIONS

Working Formulas and Tables

Frederick W. Grover

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Inductance Calculations has become a classic in its field. The work has brought to ISA's attention by the Test Measurement Division. ISA is grateful for the efforts of Jim Dorsey, Director of TMD 1980-1981.

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The author, Professor Frederick W. Grover, Ph.D. was a member of the Electrical Engineering faculty at Union College from 1920 to 1946; he had formerly been a Consulting Physicist of National Bureau of Standards.

PREFACE

The design of inductors to have a given inductance or the calculation of the inductance of existing circuits are problems of importance in electrical engineering and especially in the field of communication.

Collections of formulas for the calculation of inductance and mutual inductance for different types of coils and other inductors are to be found in various electrical engineering handbooks and notably in the publications of the National Bureau of Standards.

It has, however, been the observation of the author of the present work, who has participated in the preparation of the Bureau of Standards collections, that certain difficulties are experienced in the use of this material. The engineer who has occasion to calculate an inductance is likely to be overwhelmed by the very wealth of the formulas offered him, and especially is this true in the more common types of inductor. Furthermore, certain formulas require the use of elliptic integrals or allied functions, others zonal harmonic functions or hyperbolic functions. Other formulas appear in the form of infinite series and it is necessary to choose from among those offered that formula whose degree of convergence will best suit the problem in question. Undoubtedly these complexities discourage the computer in many cases and lead to the substitution of empirical formulas or rough approximations for the accurate formulas.

The present work has been prepared with the idea of providing for each special type of inductor a single simple formula that will involve only the parameters that naturally enter together with numerical factors that may be interpolated from tables computed for the purpose. It has been found possible to accomplish this end in all the more important cases, and, even in the more complex arrangements of conductors, to outline a straightforward procedure. For the accomplishment of this end extensive tables have had to be calculated. Fortunately, certain of the tables are useful in more than a single case, but even so the tables represent a vast amount of computation. The tabular intervals are chosen so that where possible linear interpolation or at worst the inclusion of second order differences suffice. An accuracy of a part in a thousand is aimed at in general, but for the most part the tables lead to a better precision.

Illustrative examples are included with each case and where possible the numerical values found have been checked by other known formulas or methods. Procedure for the design of the more usual types of inductor has been included.

It is believed that all the more important forms of inductor and circuit elements have been covered, but in any new case it is usual to build a formula or method from the basic formulas by the general methods that are explained in the introductory chapters.

F. W. G.

Union College,
Schenectady, N. Y.
October 1945.

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INTRODUCTION

Formulas for the calculation of self-inductance and mutual inductance are of practical importance in electrical applications. The calculation of single-layer coils, coils with rectangular cross section, current-limiting reactors, transmission lines, antennas, inductance standards may be named among the cases where a knowledge of the constants of existing circuits are required or where the problem is to design circuits to give a stated inductance.

In the absence of magnetic materials, mutual and self-inductances are parameters that are independent of the value of the current and depend only on the geometry of the system. The literature of the subject provides an abundance of formulas covering the more important cases occurring in practice but, for the most part, formulas adapted to routine calculation are not available.

For certain simple, ideal cases, exact solutions for the inductance have been found, but the expressions are complicated. For example, the expressions for circuits composed of straight filaments involve inverse hyperbolic and inverse trigonometric functions. For coaxial circular filaments, helices, and cylindrical current sheets, elliptic integrals are the normal functions. Tables of these functions are, of course, available but, in many practical computations, where these inductance formulas have to be used, the individual terms in the calculation nearly cancel, so that a high degree of accuracy must be attained in the separate terms, if the resulting calculated value of the inductance is to have even a moderate precision.

This difficulty and that of working with the more complex functions may be avoided to a great extent by the use of series developments of the exact formulas, but the computer is then met with the necessity of choosing a series that shall converge with sufficient rapidity for the problem in question. In some cases also, he is embarrassed by the wealth of series formulas available. For instance, some scores of series formulas, having different degrees of convergence, are known for the two cases of the mutual inductance of two coaxial circular filaments and the self-inductance of single-layer coils.

Furthermore, actual circuits are made up of wires, not filaments of negligible cross section, and are wound in layers or channels of rectangular cross section with insulating material between the wires. To cover practical cases, it is therefore necessary to combine solutions, which hold for the ideal cases, by methods of summation or integration in order to allow for these facts. Because of these complexities, the engineer is often deterred from making any inductance calculations at all, or is driven to the use of empirical formulas of rough accuracy and uncertain range of applicability, whose only recommendation lies in their comparative simplicity.

The present work, which is the result of years of research in this field, has for its purpose the simplification of routine calculations of mutual and self-inductances. For each case considered, so far as is possible, a single simple working formula is provided, in which appear, in addition to the given dimensions, numerical constants that may be interpolated from tables in which the shape ratios are the arguments. (Curves can, of course, be drawn from the tabular data, but the interpolation from the tables is simpler and more accurate than that obtainable from the curves.) An accuracy of a part in a thousand is in general obtainable and, in certain important cases, the results are more accurate than this. Errors in measurement of the dimensions of existing apparatus will usually be the limiting factors. Solutions of illustrative examples accompany each section of the work.

The formulas, except where otherwise stated, are for low frequencies. This does not, however, detract from their usefulness, since the effect of skin effect on inductance is small, while to take into account the effect of the capacitance of a coil on the apparent inductance, the low frequency inductance value, together with the self-capacitance of the coil, is what is required.

It is hoped that the references cited throughout the work cover sufficiently the sources of the material; much of it has not previously been published. However, no attempt has been made to present a complete bibliography of the subject. The bibliography given in Hak's *Eisenlose Drosselspulen* published by K. F. Koehler, Leipzig (1938), which is probably the most inclusive yet published, includes more than five hundred references.

Chapter 1

GENERAL PRINCIPLES

The electromotive force induced in a circuit A when the current in a circuit B is changed is proportional to the rate of change of the linkages of the flux set up by the current in B with the turns of the circuit A. If the circuits are linked through a core of iron or other magnetic material, nearly all of the flux ϕ, produced by the current, will link with the N turns of circuit A and the induced electromotive force is quite closely $-N\frac{d\phi}{dt}$. With magnetic materials, however, it is necessary to know the permeability of the material, which is a function of the magnetizing current and has to be determined by measurement for the current in question. Furthermore, although the knowledge of the permeability permits the reluctance of a complete magnetic circuit of iron to be estimated, the case of a straight magnetic core with the flux lines completed through the air is still further complicated by the difficulty of estimating the reluctance of the air path. It is, therefore, impracticable to do more than to make the roughest of calculations of the flux and therefore of the mutual inductance of circuits coupled by cores of magnetic materials. The treatment of standard apparatus employing complete magnetic circuits of iron, or circuits in which only a short air gap is included, is based on measurements of exciting current and leakage reactance.

With circuits free from iron, the case is different. The magnetic induction at any point due to current in a circuit B is directly proportional to the current i and, although the linkages of flux with the elements of a circuit A will vary, in general, from point to point, the total linkage with the circuit A is capable of being expressed as a constant M times the current. Thus, the electromotive force induced in A may be written as $-M\frac{di}{dt}$. The constant M is known as the coefficient of mutual induction or the *mutual inductance*. If the induced electromotive force is expressed in volts and the current in amperes, then M is expressed in henrys. A mutual inductance of one henry gives rise to an induced electromotive force of one volt, when the inducing current is changing at the rate of one ampere per second. For many simple circuits of only a few turns of wire, a more convenient unit of mutual inductance is the millihenry (mh), which is one thousandth of a henry, or the microhenry (μh), which is the millionth part of a henry. The latter is especially appropriate for expressing the mutual inductance of straight conductors or small coils of few turns.

The adjective "mutual" emphasizes the fact that if the electromotive force induced in circuit A by a current changing at the rate of one ampere per second in circuit B is equal to e, the same emf e is induced in circuit B when a current is made to change at the rate of one ampere per second in circuit A.

The mutual inductance may also be considered as the number of flux linkages with the circuit A due to unit current in circuit B. In the simple case where B has N_1 turns and circuit A, N_2 turns, the windings being concentrated, it is evident that the magnetic induction at any point due to unit current in B is proportional to N_1 and, therefore, the linkages with each turn of A are proportional to N_1. The total flux linkages with A, due to unit current in B are, consequently, proportional to N_1N_2. If, on the other hand, unit current is set up in coil A, the linkages with each turn of B are proportional to N_2, but there are N_1 turns in B, so that the total number of linkages with B is also proportional to N_1N_2. In general, the magnetic induction is a function of the dimensions of the inducing circuit and the number of linkages with this is a function of the dimensions of the linking circuit.

When the rôles of the two circuits are interchanged, the change in one of these factors is exactly compensated by the change in the other, and the mutual inductance is the same, whichever is the inducing circuit and whichever the circuit in which the electromotive force is induced.

The total electromotive force induced in a circuit at any moment is equal to the algebraic sum of the electromotive forces induced in the various elements of the circuit, opposing electromotive forces being regarded as of opposite signs. If we confine the consideration to frequencies such that the circuit dimensions are negligible with regard to the wave length, the magnetic induction at every point of the field is in phase with the current. In consequence, the induced electromotive forces are at all points in phase.

The total induced emf $-M\dfrac{di}{dt}$ may be considered as a summation of the elementary induced emfs around the circuit. This consideration defines what is meant by the mutual inductance of a circuit on a part of another circuit. The *partial* mutual inductance is the contribution made by the element to the *total* mutual of the circuit of which it forms a part.

Furthermore, the magnetic flux linked with a circuit element may be considered as the resultant of the fluxes contributed by the separate elements of the inducing circuit. Since, under the quasi-stationary condition assumed,

the currents in all the elements are in phase, so are the flux contributions of the separate elements. That is, the mutual inductance of an element of a circuit with the inducing circuit is the algebraic sum of the mutual inductances of the separate elements of the inducing circuit with the circuit element of the second circuit.

Assuming one circuit to be made up of elements A, B, C in series, and the other of elements a, b, c in series, the total mutual inductance is

$$M = M_{Aa} + M_{Ab} + M_{Ac} + M_{Ba} + M_{Bb} + M_{Bc} + M_{Ca} + M_{Cb} + M_{Cc},$$

and so on, for any number of sections of each circuit.

The concept of mutual inductance is not restricted to two *separate* circuits; every element of a single circuit has a mutual inductance on every other of its elements. A familiar case is presented by two coils in series carrying a current. Each coil induces an electromotive force into the other when the current is changing. In addition, the change of current induces in each coil alone an electromotive force due to the changing flux linkages of its own turns with its self-produced magnetic field. The coil is said to have *self-inductance* and the induced electromotive is commonly written $-L\dfrac{di}{dt}$, the coefficient L being designated as the *self-inductance*, or more commonly, the *inductance* of the coil. Self-inductance is merely a special case of mutual inductance. Each turn of the coil links with the magnetic field produced, not only by its own current, but by the current in the other turns also. We may, in fact, consider the self-inductance of a coil as equal to the summation of the mutual inductances of all the pairs of filaments of which the coil may be regarded as composed. Naturally, self-inductance is measured in the same unit, the henry, as is mutual inductance.

The summation principle applied to two coils A and B in series carrying the same current gives for the induced emf in the whole circuit

$$e = -L_A\frac{di}{dt} - L_B\frac{di}{dt} \mp M_{AB}\frac{di}{dt} \mp M_{BA}\frac{di}{dt}.$$

But $M_{BA} = M_{AB}$, so that

$$e = -(L_A + L_B \pm 2M_{AB})\frac{di}{dt}.$$

The inductance of the whole circuit is therefore

$$L = L_A + L_B \pm 2M_{AB}.$$

The double sign calls attention to the fact that the induced emf $-M_{AB}\dfrac{di}{dt}$ may either add to the self-induced emfs $-L_A\dfrac{di}{dt}$ and $-L_B\dfrac{di}{dt}$, or it may oppose them. By Lenz's law, the self-induced emfs are in such a direction as to oppose the change of current that gives rise to them, a fact taken into account by the negative sign. The impressed emf necessary to overcome these is, of course, $+L_A \frac{di}{dt} + L_B \frac{di}{dt}$. If the induced emf $-M \frac{di}{dt}$ adds to $-L_A \frac{di}{dt}$, the flux linkages of coil B with A add to the linkages of coil A with its own flux. The coils are said to be joined in "series aiding"; the coil A acts as though it had a self-inductance of $L_A + M_{AB}$ and the coil B as though it had an inductance $L_B + M_{AB}$. A simple interchange of the connections of one coil to the other leads to an opposing condition in which the magnetomotive force of one coil is opposed to that of the other. The apparent inductances of the coils are now $L_A - M_{AB}$ and $L_B - M_{AB}$, respectively.

When a current i_0 is established in a circuit or element of a circuit, the rise of current induces an electromotive force that opposes the rise of current. Thus, energy has to be expended by the source, in order to keep the current flowing against the induced emf. If we denote by i the current at any moment, the power expended in forcing this current against the induced emf $e = -L \frac{di}{dt}$ is $p = Li \frac{di}{dt}$. Thus the total energy supplied in raising the current to the final value i_0 is

$$W = \int_0^T Li \frac{di}{dt}\, dt = \int_0^{i_0} Li\, di = \tfrac{1}{2}Li_0^2,$$

in which T is the time interval for the establishment of the current. This energy is stored in the magnetic field and becomes available in the circuit when the current is broken. It may be shown that energy is stored in each volume element dV of the field to the amount, $\frac{H^2}{8\pi}\, dV$, where H is the magnetic field intensity at the point in question.

If while the current i_0 was being established in circuit 1 a current I_0 is maintained in a circuit 2 that has a mutual inductance M with circuit 1, then, during the rise of i an emf $-M \frac{di}{dt}$ is induced in circuit 2. To force the current I_0 against this emf, energy equal to $W_2 = \int_0^T \left(M \frac{di}{dt}\right) I_0\, dt = MI_0 i_0$ is required. If the induced emf is in such a direction that it aids the flow of the current I_0, then energy is returned to the source of I_0 and M is to be considered as negative.

The energy of a system consisting of two circuits 1 and 2, in which currents I_1 and I_2, respectively, have been established, may be calculated by supposing the current I_1 in one circuit to be made first. Then the other current is supposed to rise from zero to I_2, while I_1 is held constant. First,

with circuit 2 open, the rise of the current in circuit 1 from zero to I_1 involves the storing of energy $\frac{1}{2}L_1I_1^2$ in the magnetic field. As the current in circuit 2 then rises from zero to I_2, energy $\frac{1}{2}L_2I_2^2$ is supplied by the source 2 and, at the same time, source 1 has to supply MI_1I_2 to maintain current I_1 unchanged. The total energy of the system is, therefore,

$$W = \tfrac{1}{2}L_1I_1^2 + MI_1I_2 + \tfrac{1}{2}L_2I_2^2.$$

If there be n circuits carrying currents $I_1, I_2, \cdots I_n$, having mutual inductances M_{12}, M_{13}, etc., the energy of the whole system is the sum of terms of the form $\frac{1}{2}L_sI_s^2$, one for each circuit, and a term $M_{rs}I_rI_s$ for each pair of coupled circuits. The magnetic field intensity at each point is, of course, the resultant of the components due to the individual circuits.

Chapter 2

METHODS OF CALCULATING INDUCTANCES

1. Basic Formulas. Although the inductances and mutual inductances of circuit elements not associated with magnetic materials are independent of the value of the current and dependent only on the geometry of the system, it is only in the simplest cases that these constants can be calculated exactly. Fortunately, from these basic formulas for ideal cases, formulas applicable to the more important circuit elements met in practice may be built up by general synthetic methods. A brief survey of the methods employed in deriving the basic formulas will first be given and, following this, a treatment of methods of procedure for building up solutions of the problem for actual circuits.

(*a*) The most direct method for calculating inductances is based on the definition of flux linkages per ampere. To calculate the flux linkages, it is necessary to write the expression for the magnetic induction at any point of the field and then to integrate this expression over the space occupied by the flux that is linked with the element in question. By the Biot-Savart law, the magnetic field intensity dH, due to a current i flowing in a straight circuit element of length ds, is, at any point P of the field (see Fig. 1),

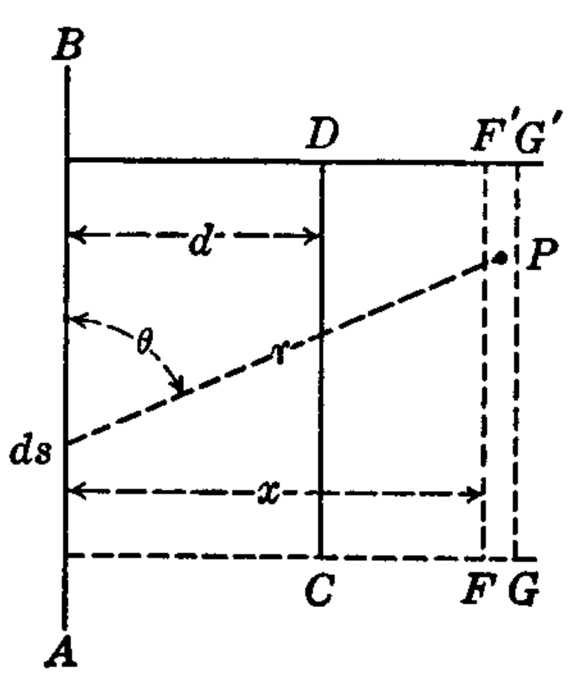

FIG. 1

$$dH = \frac{i\,ds}{r^2} \sin\theta,$$

in which r is the radius vector from ds to P and θ is the angle between the direction of ds and the radius vector. The magnetic flux is perpendicular to the plane through ds and the radius vector.

Suppose it is desired to calculate the mutual inductance of the straight parallel filaments AB and CD. The field intensity due to the current i in filament AB is found by integrating the expression for dH along the filament

AB with respect to the position of ds. This integration is readily accomplished. Since the permeability is $\mu = 1$, the flux density B at P is numerically equal to the value of H. By passing planes KK' and LL' through the ends of filament CD perpendicular to AB, the mutual inductance will be found by integrating the expression for B between these planes from CD out to infinity. The flux lines are perpendicular to the plane of the paper. Through the plane $FF'GG'$ of width dx, the total flux is found by integrating the expression for the flux density, assuming x constant. The total flux linkages are then found by a second integration with respect to x between the limits $x = d$ and $x = \infty$. Placing $i = 1$ in this expression, there results the mutual inductance of the parallel filaments. If CD and AB have the same length l, the expression found is equation (1) below.

This procedure can be extended to find the self-inductance of a straight wire of radius of cross section ρ replacing the filament AB. The linkages of flux with this are found by integrating the expression for the flux density for the space between planes through the ends of AB perpendicular to it. The expression resulting is the same as in the preceding case with ρ in place of d. This represents the flux linkages external to the wire. To this must be added the inner linkages of the wire with the flux in the cross section of the wire, in order to find the self-inductance of the straight wire of length l and radius of cross section ρ.

To find the internal flux linkages, the flux through an elementary tube of length l, radius x, and thickness dx is $l\,dx$ times the flux density. The latter is twice the current inside the tube divided by the radius of the tube. If the whole current in the wire is i, that inside the tube is $\frac{x^2}{\rho^2} i$, so that the flux $\frac{2ixl}{\rho^2}\,dx$ passes through the tube and this links with $\frac{x^2}{\rho^2}$ of the current. Integrating the weighted expression $\left(\frac{2ixl}{\rho^2}\right)\left(\frac{x^2}{\rho^2}\right) dx$ with respect to x between zero and ρ, there are found the total flux linkages with current. Equating this to $L_i i$, the internal self-inductance L_i is found, which, added to the external inductance, gives the self-inductance of the straight wire.

(*b*) The internal inductance of the straight wire may be found also by noting that inside the conductor the flux density B is $\frac{2ix}{\rho^2}$. The volume element is $2\pi xl\,dx$. The energy in this volume is $\frac{B^2}{8\pi}(2\pi xl\,dx) = \frac{i^2x^3l}{\rho^4}\,dx$. Integrating this with respect to x between zero and ρ, the energy inside the conductor is found, and equating this to $\frac{1}{2}L_i i^2$, the same value $L_i = \frac{l}{2}$ is found as before.

In general, this latter method does not work out as simply as the use of the Biot-Savart law, or the Neumann method, which follows.

(*c*) The Neumann formula for the mutual inductance of two circuit elements is given by

$$M = \iint \frac{\cos \epsilon}{r} \cdot ds\, ds', \tag{a}$$

in which ϵ is the angle of inclination between the two circuit line elements ds and ds', r is the radius vector between them, and the integration is to be taken over the contours of the two circuit elements.

This is the most general expression for finding the mutual inductance. It leads quite simply to a *formal* expression for the mutual inductance even though for most cases it is not possible to perform the integrations. However, in such cases also it is possible to obtain a numerical value for a specified case by mechanical integration, using Simpson's rule or the like. Naturally, however, this calculation may be tedious.

The mutual inductance of two parallel filaments may of course be found by the Neumann formula, but not as simply as by use of the Biot-Savart law. For inclined filaments, however, the Neumann formula has the advantage, and the formula for the mutual inductance of two straight filaments placed in any desired position has also been obtained by its use. This latter is a closed formula consisting of terms involving inverse hyperbolic and inverse trigonometric functions.

Other very important ideal cases solved by its use are the mutual inductance of two coaxial circular filaments,[1] the self-inductance of a helix,[2] the mutual inductance of a helix (or cylindrical current sheet) and a coaxial circle,[3] and the mutual inductance of two coaxial cylindrical current sheets.[4] For these cases elliptic integrals are the normal functions. For the coaxial circles and the solenoid, complete elliptic integrals of the first and second kinds appear; for the other cases elliptic integrals of all three kinds.

Calculations by means of these basic formulas depend upon tables of functions which are not difficult to use. Cases are often encountered, however, where the positive terms in the equations are nearly canceled by the negative, and care must be taken that the individual terms are calculated with a degree of accuracy sufficient to lead to the desired accuracy in the result. In other cases it is difficult to interpolate accurately the value of the elliptic integral. For many purposes, therefore, series expansions of the basic formulas are to be preferred to the basic expressions themselves. By their means, calculations can be made without recourse to special tables, and it is only necessary to select from those available a series formula whose convergence is satisfactory for the case in question.

A further advantage of the series expansions lies in their suitability for purposes of integration. It must be remembered, however, that no single

[1] References are numbered consecutively throughout the book and are found in a section beginning on page 283 at the end of the book.

series expansion can replace the basic formula. Some series converge well for small values of the variable, others for large values of the variable, while for intermediate cases no series may be entirely satisfactory. A further difficulty lies in the confusion caused by the large number of such expansions that have been found. For example, for the calculation of the mutual inductance of coaxial circles there are available, in addition to the basic elliptic integral formula, other elliptic integral formulas to other moduli, series in powers of the modulus, series in powers of the complementary modulus, series involving the shape ratio parameters, arithmetical-geometric mean series and q series developments. (For further details, see reference 5.)

(*d*) The list of basic cases already detailed includes those fundamentally most important. However, by integration of series developments of these cases, term by term, series expressions for other important cases are possible. For example, series expansions for the mutual inductance of cylindrical current sheets may be derived by the integration of a series expression for the mutual inductance of coaxial circles. Also a formula for the mutual inductance of circles with parallel axes may be derived from the coaxial case as a series involving zonal harmonics,[6] and a series for the mutual inductance of solenoids with parallel axes may be obtained from this by integration over the lengths of the two solenoids.[7] In all these cases it is necessary to select for integration a series in which the terms involve the variable of integration (a length, for example) directly.

2. Formulas for Actual Circuits and Coils. The basic formulas apply to the ideal cases of straight, circular, or helical filaments or to cylindrical current sheets of negligible thickness. Actual circuits are composed of conductors of appreciable cross section, of single-layer windings, of windings in channels of rectangular cross section. Not only is the current distributed over a finite cross section of conducting material, but the wires are separated by insulating spaces. Formulas for the inductance of actual circuit elements have to be derived by correcting the ideal basic formulas or by building up solutions by combinations of the formulas that hold for the ideal cases. Several fundamental methods are available for accomplishing these ends.

(*a*) *Integration of Basic Formulas over the Cross Section of the Winding.* Such direct integration is, in general, too difficult, but in a few instances has been accomplished, notably for the inductance of a circular coil of rectangular cross section, where the basic formula employed is that for the mutual inductance of coaxial circular filaments. Likewise, by integrating formulas for the mutual inductance of coaxial cylindrical currents the case of thick coaxial solenoids[8] has been treated. Also, from the basic formula for circles with parallel axes has been derived a formula for the mutual inductance of thick coils with parallel axes.[9] It should be noted that these formulas suppose the current to be uniformly distributed over the cross section, that is, no account is taken of the insulating space between the wires. Methods for applying this correction are given below, but fortunately the correction is of small importance, unless the wire has a large cross section and the insulation is thick.

(b) *Taylor's Series Expansions.* The mutual inductance of two windings having appreciable cross sectional dimensions may be referred to that of the central filaments of the coils by the use of a Taylor's series expansion around the positions of these filaments. Although the method, in principle, applies to cross sections of any shapes, the formula below applies only for rectangular cross sections. If N_1 and N_2 represent the numbers of turns on the two coils and M_0 is the mutual inductance of the central filaments OO' and PP', Fig. 2, then to a first approximation the mutual inductance of the coils is

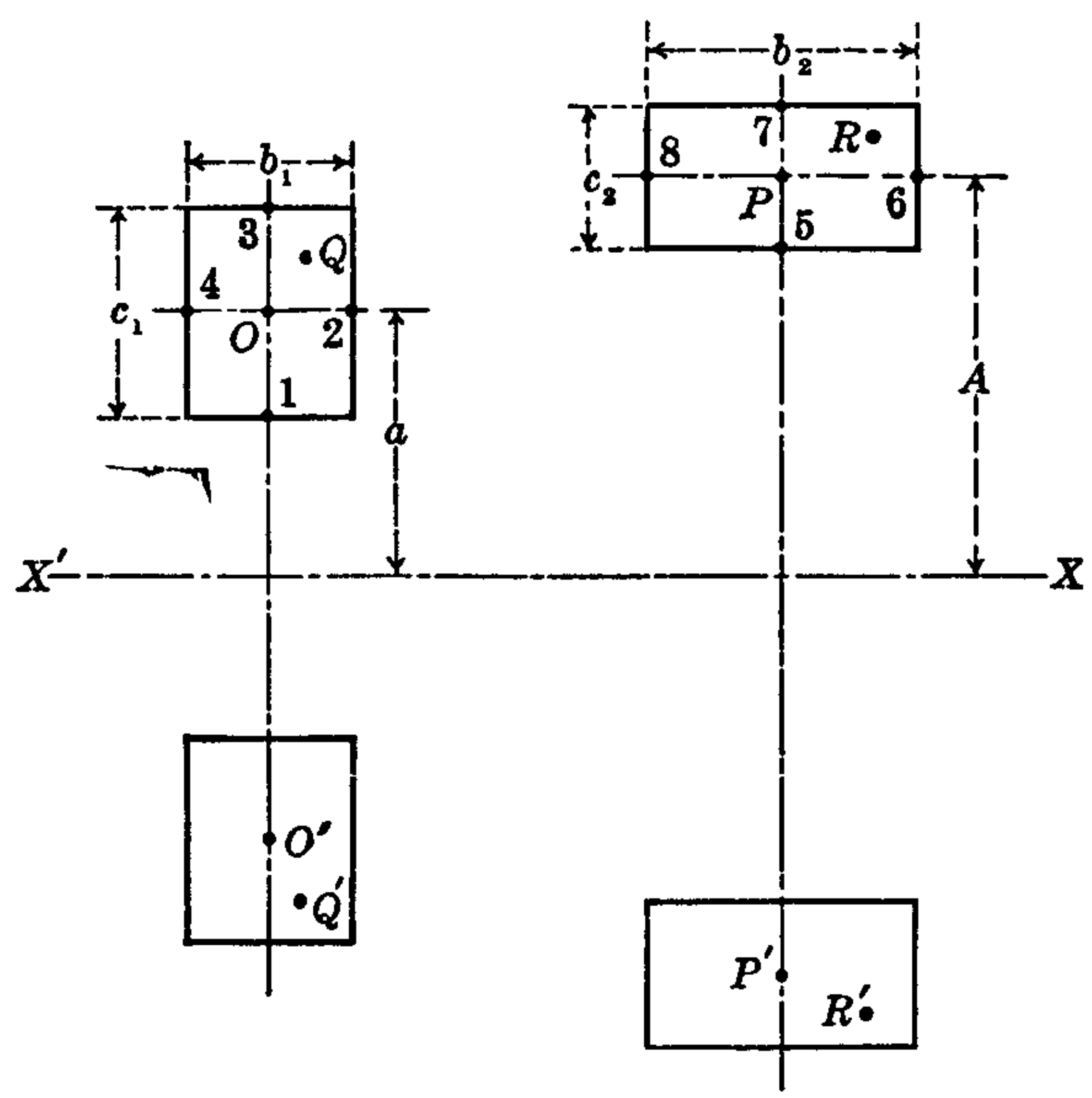

FIG. 2

$N_1N_2M_0$. Referring the coordinates of any filament QQ' to the center O of the section as origin, the mutual inductance of filament QQ' on the central filament PP' of the other coil may be expressed in a Taylor's series referred to M_0, in which the differential coefficients are the values that hold at the central points of the sections.

If this series be averaged over the cross section by integrating over the section and dividing by the area of the cross section, there is found the value of the mutual inductance (divided by N_1N_2) of the left-hand coil of Fig. 2 and the central filament PP' of the other coil. If this value be denoted by M_1, the process may be repeated by referring the position of any filament RR' of the second coil to that of PP', and expressing the mutual inductance of RR' and the left-hand coil in a Taylor's series referred to M_1. Averaging this over the cross section of the second coil there results a value M_2, and the mutual inductance of the two coils of rectangular cross section is $M = N_1N_2M_2$.

The integrations are readily performed, since the differential coefficients are constants, that is, functions of the position of the central points to which they refer. This integration has been carried out to include differential coefficients of sixth order,[10] but the expressions become long and cumbersome. The expression including differential coefficients of second order only is

$$M = N_1N_2\left[M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d^2M_0}{dx^2} + c_1^2\frac{d^2M_0}{da^2} + c_2^2\frac{d^2M_0}{dA^2}\right\}\right]. \quad \text{(b)}$$

The calculation is thus made to depend upon the basic formula for the mutual inductance of the central filaments, the cross sectional dimensions, and the differential coefficients. The latter are obtained by differentiating the general basic formula for two filaments and then substituting the radii and axial distance corresponding to the central filaments.

This method has been used to find very accurate formulas for equal coaxial coils of rectangular cross section.[10] Its employment is not, however, restricted to coaxial elements, nor is it necessary that circular filaments be postulated. The closer the circuits and the greater the cross sectional dimensions, the more important are the terms involving higher order differential coefficients, but the work of Rosa [10] has shown that unfavorable cases require usually the inclusion of no order higher than the sixth.

These formulas assume that the current is uniformly distributed over the cross sections of the coil. The correction is negligible for windings of wire of sizes usual in practice. The special case where one or both of the cross sections has one of its cross sectional dimensions zero (simple layer winding) is also treated by the formulas derived above.

(c) *Rayleigh Quadrature Formula.* Rayleigh [11] has given another form to the expression derived for coils of rectangular cross section by the Taylor's series expansion. In this, the mutual inductance of the coils is made to depend upon an average of the mutual inductances of certain chosen filaments. Referring to Fig. 2, Rayleigh's expression is

$$M = \tfrac{1}{6}(M_{P1} + M_{P2} + M_{P3} + M_{P4} + M_{O5} + M_{O6} + M_{O7} + M_{O8} - 2M_{OP}), \quad \text{(c)}$$

where M_{P1} is the mutual inductance of filaments $11'$ and PP', M_{O5} is the mutual inductance of filaments OO' and $55'$, etc. Thus the calculation of the mutual inductance of the coils is reduced to finding the average of the values for certain filaments, each one of which may be calculated by the basic formula. With suitable tables this is easily accomplished. This formula is known as the Rayleigh quadrature formula. Its accuracy is, of course, limited by the sufficiency of the terms involving differential coefficients of second order in the Taylor's series from which it is derived. The quadrature formula is useful in any case where rectangular (or straight line) cross sections are involved, and the basic filament formula is available.

(*d*) *Lyle Method of Equivalent Filaments.* The use of the Rayleigh formula virtually replaces the coils of rectangular cross section by certain specially chosen filaments. A still more striking example of this method of solution is afforded by the Lyle method [12] of equivalent filaments. This is very accurate for coaxial coils of dimensions such that fourth order and higher order differential coefficients in the Taylor's expansion are negligible. The dimensions of the equivalent filaments in any case is illustrated by Fig. 3, which shows

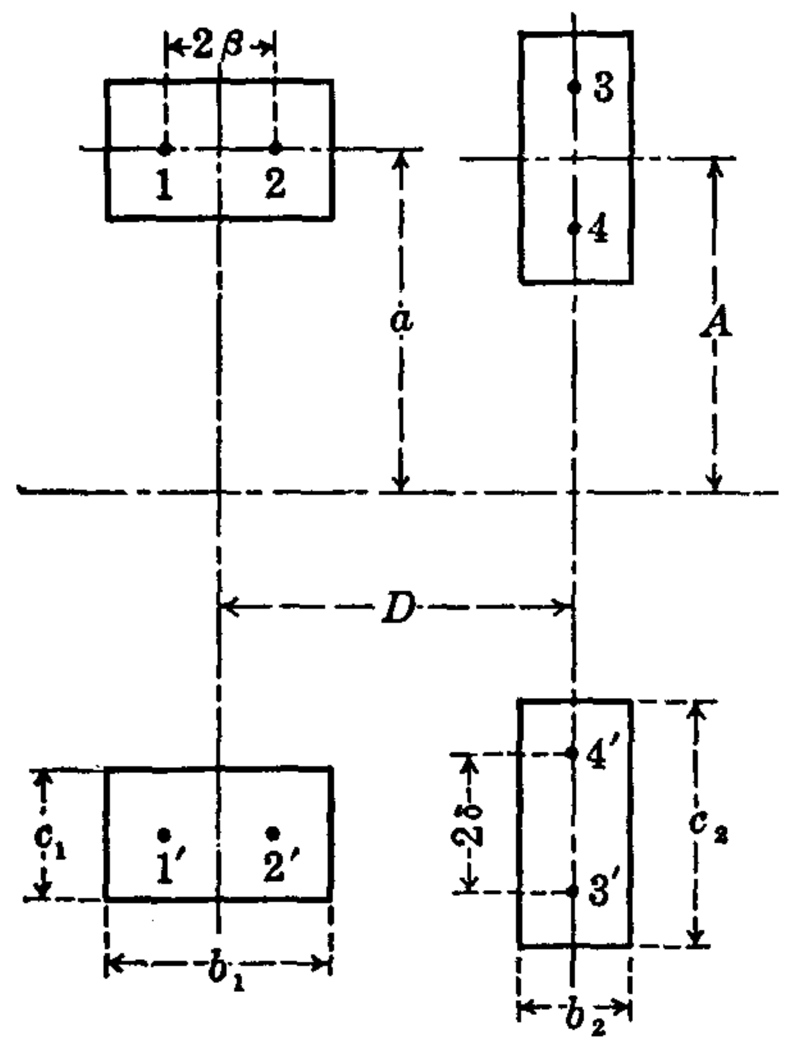

Fig. 3

two circular coils of rectangular cross section of mean radii a and A, axial dimensions b_1, b_2, radial dimensions c_1 and c_2 and spacing of median planes D.

Lyle's method replaces the left-hand coil $b_1 > c_1$ by two filaments 11′ and 22′ of radii $r_1 = a\left(1 + \dfrac{c_1^2}{24a^2}\right)$, spaced on each side of the median plane at a distance β, which is given by $\beta^2 = \dfrac{b_1^2 - c_1^2}{12}$.

The right-hand coil in Fig. 3 is replaced by two filaments 33′ and 44′ lying in the median plane having radii of $r_2 + \delta$ and $r_2 - \delta$, respectively, where $r_2 = A\left(1 + \dfrac{b_2^2}{24A^2}\right)$ and $\delta^2 = \dfrac{c_2^2 - b_2^2}{12}$.

Thus to calculate the mutual inductance of the coils of Fig. 3, the basic formula for the mutual inductance of coaxial circles is applied to find the mutual inductance of four pairs of filaments, each filament assumed to have half the turns of its coil. The mutual inductance of the coils is the sum of the four values. The following scheme shows the arrangement of the calculation:

Filaments	Product of Turns	Radii	Axial Spacing
11′ and 33′	$\dfrac{N_1N_2}{4}$	r_1 and $(r_2 + \delta)$	$D + \beta$
11′ and 44′	$\dfrac{N_1N_2}{4}$	r_1 and $(r_2 - \delta)$	$D + \beta$
22′ and 33′	$\dfrac{N_1N_2}{4}$	r_1 and $(r_2 + \delta)$	$D - \beta$
22′ and 44′	$\dfrac{N_1N_2}{4}$	r_1 and $(r_2 - \delta)$	$D - \beta$

A special case in the use of Lyle's method is that of a square cross section. For such a section β and δ are zero, so that the coil is replaced by a single filament of radius $r = a\left(1 + \dfrac{b^2}{24a^2}\right)$, where a and b are the mean radii of the coil and the side of the section, respectively.

(*e*) *Sectioning Principle.* Coils of large cross sectional dimensions, compared with their spacing and radii, require for accuracy the inclusion of higher order differential coefficients in the Taylor's series expansion, so that the Rayleigh formula and the Lyle method of equivalent filaments give only approximate values. The errors in practical cases are unimportant, especially since it is difficult to measure the dimensions and spacing of such coils with accuracy. However, in precision work, the errors of these two formulas may be reduced by sectioning the coils and applying the formulas of Rayleigh or Lyle to the individual sections. Thus, for example, in Fig. 3, each coil may be imagined to be divided into two sections giving coils P, Q and R, S. By the summation principle the total mutual inductance of the coils is then $M_{PR} + M_{PS} + M_{QR} + M_{QS}$. Each of the four terms may be calculated by replacing each section by equivalent filaments, or each pair of sections may be treated by the Rayleigh formula. Because of the decreased cross sections, the errors due to the neglect of higher order terms are much reduced. This process may be extended to a greater number of sections, but it is evident that thereby the number of pairs of filaments to be calculated increases rapidly. For example, if each coil in Fig. 3 is divided into two sections, each coil will be replaced by four filaments in the Lyle method, each carrying a quarter of the number of the turns of the coil. Sixteen pairs of filaments would have to be calculated.

By application of the summation principle, the mutual inductance of coils of large cross section in contact, or nearly in contact, may be found by basing the calculation on formulas for the *self*-inductance of coils.[13] For example, if it is desired to calculate the mutual inductance M_{AB} of the two coils A and B of Fig. 4, which have the same mean radius of winding and the same radial dimension c but different axial dimensions b_1 and b_2, the procedure is as follows: Imagine a coil C of rectangular cross section just filling in the space between A and B. Supposing a winding density of N turns per unit area of cross section, the mutual inductance of A and B under that assumption would be

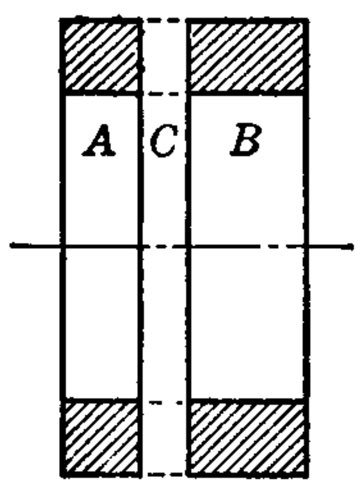

FIG. 4

$$2m_{AB} = (L_{ABC} + L_C) - (L_{AC} + L_{BC}), \qquad \text{(d)}$$

in which L_{ABC} is the self-inductance of the three coils in series, aiding, L_{AC} the self-inductance of A and C in series, aiding, etc. Each of the terms in the formula would be calculated from a formula for the self-inductance of a circular coil with rectangular cross section. The value m_{AB} gives the mutual inductance of the coils A and B under the assumption that they have NA_1 and NA_2 turns, respectively, $A_1 = b_1c$ and $A_2 = b_2c$ being the areas of the cross sections. If the actual numbers of turns of the coils are N_1 and N_2, then the actual mutual inductance of the coils is $M_{AB} = \left(\frac{N_1}{NA_1}\right)\left(\frac{N_2}{NA_2}\right) m_{AB}$. The quantities $\frac{N_1}{A_1}$ and $\frac{N_2}{A_2}$ are the actual winding densities on the coils, so that the ratio $\frac{M_{AB}}{m_{AB}}$ is the ratio of the product of the true winding densities to the square of the assumed density N. If N is taken equal to unity, M_{AB} is found by multiplying the value m_{AB} so calculated by the product of the actual winding densities.

The foregoing principle is applicable also to single layer coils and has been found useful in the solution of other problems where coils of large cross section are placed near together.[13]

(*f*) *Geometric Mean Distance Method.* The calculation of the mutual inductance of straight conductors of rectangular cross section may be treated by the Taylor's series expansion method or by the Rayleigh formula. A more useful method and one applicable to other shapes of cross section is the geometric mean distance method, treated below. This is essentially a method of replacing the conductors by equivalent filaments, so that the basic formulas for straight filaments or the series expressions for coaxial filaments close together can be used directly.

(*g*) *Correction for Insulating Space.* The preceding methods for calculating the inductances of windings of appreciable cross sections have assumed that the current is uniformly distributed over the cross section of the winding. This assumption leads to no sensible error in the calculation of the mutual inductance of ordinary windings, except for coils in contact. Self-inductance formulas need correction, especially where heavy conductors and thick spacing of insulation are employed. For calculating the correction the simplest and most accurate method proposed is that of Rosa.[14] The Rosa method may best be made clear by an example. Suppose we have a helical single-layer winding, Fig. 5, wound with round wire in a pitch p. The basic formula for the inductance of a single-layer coil or helix is that for a cylindrical current sheet. Suppose this to consist of a winding of metallic tape of negligible thickness with a number of turns such that each turn of round wire of the winding is at the center of a turn of tape. The turns of tape are supposed to be separated by insulating spaces of negligible thickness. If the winding has N turns, the length of the cylindrical current sheet is Np. This is, for the coil, the equivalent current sheet; evidently it will project beyond the winding slightly at both ends. The inductance of the coil is to a first approximation the same as that of the equivalent current sheet, which is calculated by the basic formula.

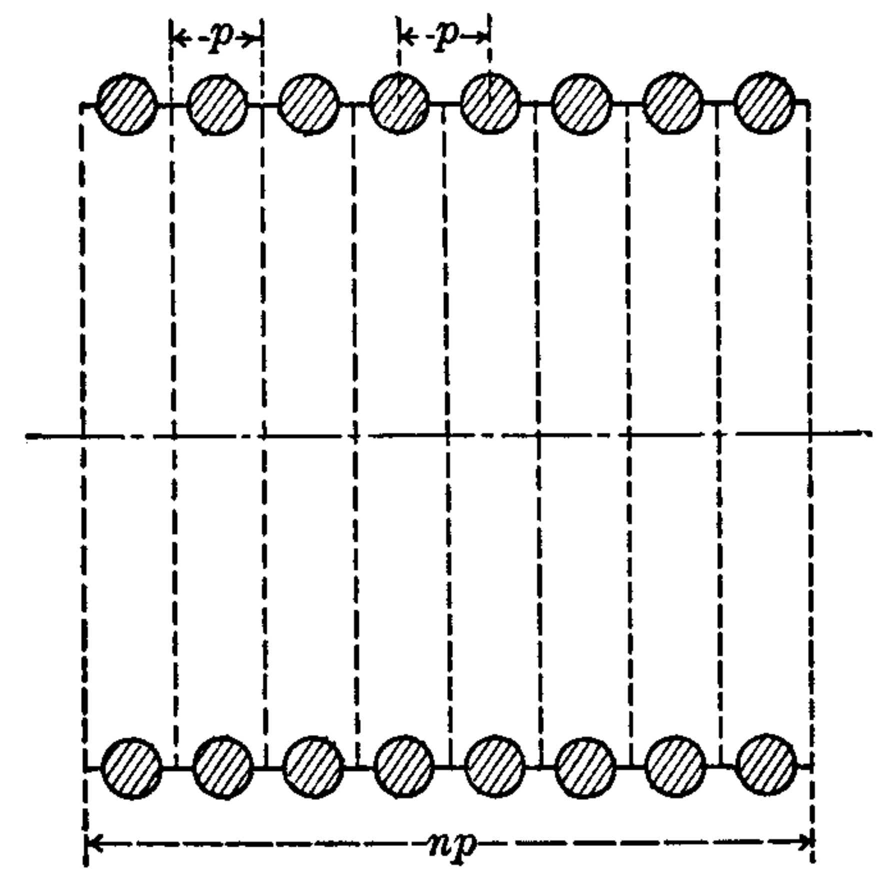

FIG. 5

The inductance of the coil differs from that of the equivalent current sheet for two reasons:

1. The self-inductance of a turn of the wire is slightly different from that of a turn of the current sheet. The two turns have the same mean radius but different cross sectional shapes and areas. This small difference may be accurately calculated by use of geometric mean distance formulas, and the